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Fluctuations in fragmentation processes

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Abstract. We investigate the role of fluctuations in fragmentation processes using a simple analytic method to verify the stability of the mean-field similarity solution. For the fragmentation process which slows down with decreasing fragment size, the cascade becomes self-averaging in the late stages of branching. For the fragmentation which speeds up with decreasing fragment size, fluctuations dominate. In both cases the deviation from the mean-field solution is neither Gaussian nor symmetric. The steady-state regime in the presence of an external source is investigated and compared with the above results. Some exact solutions are obtained for the case of homogeneous fragmentation and for the discrete Yule–Ferry cascade process of decay into two equal fragments. The connection of the kinetic method with the combinatorial-type analysis of branching is indicated. Applications are made to the process of phonon decay.

1. Introduction

In what follows we address some problems related to the role of fluctuations in cascade-like phenomena [1, 2]. We investigate the stability of the mean-field solutions in fragmentation [2, 3] with respect to fluctuations, develop the picture of the steady-state regime of fragmentation, and solve a sample problem to indicate a possible mapping between the combinatorial description [4–6] of the tree process and the kinetic approach. We also discuss the physical realization of the cascade process by studying phonon decay [7, 8].

In cascade-like phenomena the appearance of self-averaging (and the time required for this) or, in contrast, the appearance of non-self-averaging (and the amplitude of fluctuations) have not been addressed so far in detail. We think there is a need for a quantitative description. This is the motivation of the present work.

For brevity we denote by the phenomena of fragmentation an inter-disciplinary approach for describing many different physical systems. Generally speaking, it includes any kinetic process with scattering, decay, breaking, or splitting of an initial state into two (or several) new states (fragments), etc. A characteristic feature of any cascade process is that the fragments then proceed independently and in the same fashion, i.e. decay to smaller pieces. This is therefore a linear system with the generic feature known as ultrametricity [9]. We may cite phonon down-conversion

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in solid state physics [7], particle decay in nuclear physics [10], radiation disrupture of condensed matter [6], cosmic showers [11], atomic collision cascades [12], neutron noise in a multiplying medium [11], electron transport and even cell division [11] as examples.

It is rather obvious that fluctuations are inherent in any fragmentation process: for any given realization the probability of the distribution over fragment sizes, say, at some late stage of the process is the result of the repetition of numerous breakups, each of them being weighted with the cross-section of the single act of decay. If this cross-section is not somehow peaked in a deterministic way—if each decay corresponds to the breakup into two fragments of comparable size (energy)—the probability distribution with a significant width would be predetermined just by the spread of the very first decay act.

Thus, the important issue is related to the information that one can extract from the study of the decay products of single species. We address this problem in section 2. The averaging over many realizations is formulated in section 3 as the steady-state regime. The possibility of having a steady-state solution is not obvious from the very beginning, since the breakup probability may depend on the current fragment size (energy) in two qualitatively different manners: the decay may either slow down or speed up with diminishing fragment size. Here the discrete model of the Yule–Ferry cascade [11] reconsidered by some of us [13] proves to be useful for understanding the convergence of the probability distribution to the steady state. In section 4 we discuss the correspondence between the combinatorial approach along a tree and the kinetic process developing along the same tree. For the simple discrete model [5, 6] the one-to-one equivalence can be found, unlike the case of more complicated situations. In section 5 we apply the formalism of section 2 to the process of phonon frequency-down conversion in the simplest homogeneous case. Section 6 briefly summarizes the results.

2. Fragmentation

In this section we consider the typical example of the decay process—the fragmentation model by Kolmogorov–Filippov studied in [1–3]. Let us introduce the kernel $K(x,y)$ describing the rate density of splitting of the initial state y into two fragments of sizes x and $y-x$; the total mass (energy) is thus conserved. We define the conditional probability (with continuous argument x) $W(N,x,t|1,x_0,t_0)$ of having N fragments of size x at time t given there was one initial 'fragment' of size x_0 at time t_0. For the case of binary fragmentation (see [14] and appendix B) the kernel $K(x,y)$ in [2, 3] was taken to have the form

$$K(x,y)=y^{\beta} \tag{2.1}$$

and the so-called mean-field solution was obtained, i.e. the Boltzmann equation for the first moment

$$\langle N(x,x_0,t)\rangle=\sum_N NW(N,x,t|1,x_0,0)$$

i.e. the equation

$$\partial_t\langle N(x,x_0,t)\rangle=-\langle N(x,x_0,t)\rangle\int_0^{x_0}\mathrm{d}y\,K(y,x_0)+2\int_x^{x_0}\mathrm{d}y\,K(y,x_0)\langle N(x,y,t)\rangle \tag{2.2}$$

was solved analytically in terms of the Kummer function

$$\langle N(x,x_0,t)\rangle = \mathrm{e}^{-tx_0^{\beta+1}}\left[\delta(x-x_0) + 2tx_0^{\beta}\ {}_1F_1\left(\frac{\beta+3}{\beta+1};2;(x_0^{\beta+1}-x^{\beta+1})t\right)\right]. \quad (2.3)$$

The dependence of kernel (2.1) upon β gives the relaxation rate of mass (energy) y to be

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{y}{\tau(y)} = -y\int_0^y \mathrm{d}x\, K(x,y) = -y^{\beta+2}$$

and describes two situations: at $\beta > -1$ the relaxation is infinitely long, while at $\beta < -1$ the fragmentation process speeds up and completes in finite time of the order of the very first breakup time. The latter results in a 'shattering' kinetic transition—part of the initial mass (energy) starts (formally) to accumulate at the low limit, $x \to 0$.

The solution (2.3) has a simple asymptotic form [2]. Above shattering, at $\beta > -1$ and $x < x_0,\ tx_0^{\beta+1} \gg 1$, it reduces to

$$\langle N(x,x_0,t)\rangle = \frac{2x_0}{\Gamma[(\beta+3)/(\beta+1)]}t^{2/(\beta+1)}\mathrm{e}^{-tx^{\beta+1}} \quad (2.4)$$

(the so-called similarity solution). Below shattering, at $\beta < -1$ and $x < x_0$, $tx^{\beta+1} \gg 1$ we have

$$\langle N(x,x_0,t)\rangle = \frac{2x_0^{\beta}x^{-\beta-3}}{\Gamma[(\beta-1)/(\beta+1)]}t^{-2/(\beta+1)}\mathrm{e}^{-tx_0^{\beta+1}}. \quad (2.5)$$

The immediate question arises whether this solution is stable with respect to fluctuations, as has been said above. In other words, the issue is whether the solutions (2.3)–(2.5) can be seen in a single experiment. This question has been addressed mathematically by Filippov [2]. We shall use a different method which allows us to derive equations for the entire distribution and its moments and obtain some explicit solutions together with their physical meaning. To the best of our knowledge the method employed goes back to unpublished works by Landau on cosmic showers (at the end of the 1940s) but has, since that time, been re-discovered by other researchers [12] (there exist other methods making use of more standard approaches [11]). In the method which we use one writes down the explicit equation for $W(x,x_0,t|1,x_0,0)$ in the form

$$\begin{aligned}\partial_t W(N,x,t|1,x_0,0) &= -W(N,x,t|1,x_0,0)\int_0^{x_0}\mathrm{d}y\,K(y,x_0-y)\\ &+\sum_{N'}\int_x^{x_0-x}\mathrm{d}y\,K(y,x_0)W(N',x,t|1,y,0)W(N-N',x,t|1,x_0-y,0)\\ &+\int_0^x \mathrm{d}y\,K(y,x_0)W(N,x,t|1,x_0-y,0)\\ &+\int_{x_0-x}^{x_0}\mathrm{d}y\,K(y,x_0)W(N,x,t|1,y,0).\end{aligned} \quad (2.6)$$

which makes use of the fact that a shower (cascade) can be described in terms of the very first act of decay and can be derived from the backward Chapman–Kolmogorov equation (see appendix A). (The linear terms take into account the situation when only the larger fragment can contribute to $W(N, x, t|1, x_0, 0)$ at given $x, x < x_0$.)

The corresponding equation for the generating function

$$G(x, x_0, t; z) = \sum_N W(N, x, t|1, x_0, 0) \exp(zN) \tag{2.7}$$

reads

$$\begin{aligned}\partial_t G(x, x_0, t; z) = &-G(x, x_0, t; z) \int_0^{x_0} \mathrm{d}y\, K(y, x_0) \\ &+ \int_x^{x_0 - x} \mathrm{d}y\, K(y, x_0) G(x, y, t; z) G(x, x_0 - y, t; z) \\ &+ \int_0^x \mathrm{d}y K(y, x_0) G(x, x_0 - y, t; z) \\ &+ \int_{x_0 - x}^{x_0} \mathrm{d}y\, K(y, x_0) G(x, y, t; z)\,.\end{aligned} \tag{2.8}$$

x, x

The moments can be defined via

$$\langle N^k(x, x_0, t)\rangle = \partial^k G(x, x_0, t; z)/\partial z^k{}_{|z=0} \tag{2.9}$$

and the cumulants can be determined by the connection formulae, the first three of which are

$$c_1 = \langle N\rangle \qquad c_2 = \langle N^2\rangle - \langle N\rangle^2 \qquad c_3 = \langle N^3\rangle - 3\langle N^2\rangle\langle N\rangle + 2\langle N\rangle^3\,. \tag{2.10}$$

From (2.8) the equation for the second moment is

$$\begin{aligned}\partial_t \langle N^2(x, x_0, t)\rangle = &-\langle N^2(x, x_0, t)\rangle \int_0^{x_0} \mathrm{d}y\, K(y, x_0) \\ &+ 2\int_x^{x_0} \mathrm{d}y\, K(y, x_0)\langle N^2(x, y, t)\rangle \\ &+ 2\int_x^{x_0 - x} \mathrm{d}y\, K(y, x_0)\langle N(x, y, t)\rangle\langle N(x, x_0 - y, t)\rangle\,.\end{aligned} \tag{2.11}$$

Higher equations can easily be written.

Comments should be made about the normalization in the continuum limit. Since the probability W is dimensionless together with all its moments, the continuum limit can be conveniently defined through the discrete version if we express all the masses in terms of the unit mass step of the discrete formulation. Then the increase of the number of discrete states affects the value of x_0 and defines the continuum limit. This means that all powers of δ-functions should be considered as Kronecker symbols.

To investigate fluctuations, we solve (2.11) with the kernel (2.1) using the solution (2.3) for $\langle N(x,x_0,t)\rangle$ as the Green function:

$$\langle N^2(x,x_0,t)\rangle = \langle N(x,x_0,t)\rangle + 2\int_0^t \mathrm{d}t' \int_{2x}^{x_0} \mathrm{d}y\, y^\beta \langle N(y,x_0,t')\rangle \times \int_x^{y-x} \mathrm{d}z \langle N(x,z,t-t')\rangle\langle N(x,y-z,t-t')\rangle\,. \tag{2.12}$$

At arbitrary β the corresponding integrals for the Kummer function are not known analytically, and the only answer can be written in terms of multiple sums, which is not useful for further investigation. However, in the case $\beta = 0$ the exact answer is given in appendix C. Asymptotical behaviour is studied below. Namely, some progress can be made if we confine ourselves to the region of validity of (2.4), (2.5). Fluctuations are different above and below the shattering transition $\beta = -1$.

2.1. Above shattering, $\beta > -1$

We return to (2.11) and use (2.4) to evaluate the inhomogeneity under conditions leading to the similarity solution (2.4) at $x \ll x_0$. The last term in (2.11) gives

$$\frac{4}{3\Gamma^2[(\beta+3)/(\beta+1)]^2} x_0^{\beta+1} t^{4/(\beta+1)} \mathrm{e}^{-2tx^{\beta+1}}\,. \tag{2.13}$$

The solution of (2.11) can be found to be in the form $\langle N^2(x,x_0,t)\rangle = x_0^2 f_0(x,t)$. Plugging this ansatz into (2.11) one can see that it corresponds to the quasi-steady regime ($\partial_t f_0 \approx 0$) and finally in the main order

$$\langle N^2(x,x_0,t)\rangle = \langle N(x,x_0,t)\rangle^2\,. \tag{2.14}$$

This result can be directly found from (2.12): one notices that $t' \ll t$, then integration over time can be performed with the first Kummer function, while the other two are to be replaced by the similarity solutions. Equation (2.14) suggests that the similarity solution corresponds to effective averaging of fluctuations. One can extend this statement by showing that to the same order the generation function is simply

$$G(x,x_0,t;z) = \exp\left\{z\langle N(x,x_0,t)\rangle\right\}\,. \tag{2.15}$$

This can be done term by term after expanding the exponential (2.15) into series and using (2.8). However, to this order, there is no information whatsoever about the fluctuation amplitude, since all the cumulants above the first one are equal to zero.

One can therefore compare then two different sources of corrections to (2.14). The first correction comes from the next order in the expansion of the Kummer function and exact limits of integration in (2.11). Denoting the similarity solution (2.4) as $\langle N_0(x,x_0,t)\rangle$ we obtain

$$\langle N(x,x_0,t)\rangle - \langle N_0(x,x_0,t)\rangle = \langle N_0(x,x_0,t)\rangle \left[\frac{\beta-1}{\beta+1}\left(\frac{x}{x_0}\right)^{\beta+1} + \frac{2(1-\beta)}{(\beta+1)^2}\frac{1}{tx_0^{\beta+1}}\right]\,. \tag{2.16}$$

Comparison of two terms on the right-hand side of (2.16) gives two different cases $\beta > 1$ and $-1 < \beta < 1$. These cases can be traced further in (2.11) with explicit account taken for the integration limits. In the former case one can use the ansatz

$$\langle N^2(x, x_0, t)\rangle = \langle N_0(x, x_0, t)\rangle^2\left[1 + \left(\frac{x}{x_0}\right)^2 f_>(x, t)\right] \tag{2.17}$$

which results in

$$f_>(x, t) = \frac{2}{\beta+1}\left[\beta - 1 + \frac{4}{(\beta+1)tx^{\beta+1}}\right]. \tag{2.18}$$

The latter case $-1 < \beta < 1$ requires the ansatz

$$\langle N^2(x, x_0, t)\rangle = \langle N_0(x, x_0, t)\rangle^2\left[1 + \left(\frac{x}{x_0}\right)^{\beta+1} f_<(x, t)\right] \tag{2.19}$$

and gives

$$f_<(x, t) = \frac{2(\beta-1)}{(\beta+1)^2}\left[1 - \frac{2}{(\beta+1)tx^{\beta+1}}\right]. \tag{2.20}$$

The second source of corrections is due to the fluctuations caused by the initial condition, which we have seen in the right-hand side of (2.12). It is simply equal to $\langle N(x, x_0, t)\rangle$. Comparing both types of corrections one can easily check that the second type dominates. This can be traced for higher moments, and we get

$$\langle N^n(x, x_0, t)\rangle = \langle N_0(x, x_0, t)\rangle^n + \langle N(x, x_0, t)\rangle. \tag{2.21}$$

Fluctuations can be conveniently characterized by the signal-to-noise ratio, $c_1/c_2^{1/2}$, which to our approximation is just the square root of the mean-field solution. Therefore, near the maximum of the mean-field solution at $t_m = 2x^{-\beta-1}/(\beta+1)$, we have $c_1/c_2^{1/2} \sim x_0 x^{-2}$. This ratio is large with respect to unity and indicates that eventually the cascade enters into a self-averaging regime. Computer simulation of the system considered here shows that numerically this averaging takes place at a rather late stage (or small enough x), where there appears a 'window' with $\langle N(x, x_0, t)\rangle \gg 1$. Outside of the window fluctuations dominate. From figure 1 we can see how small are the corresponding fragment sizes x. The third cumulant which describes the asymmetry of the distribution near $\langle N(x, x_0, t)\rangle$ (sometimes we shall refer to this asymmetry as a non-Gaussian distribution) follows from (2.21) and to the main order is equal to $c_3 = \langle N_0(x, x_0, t)\rangle - 3\langle N_0(x, x_0, t)\rangle^2$. At small and large times this difference is positive and the ratio $c_3^{1/3}/c_2^{1/2}$ is as large as $\langle N_0(x, x_0, t)\rangle^{-1/6}$. The third cumulant passes through zero near the edges of the window and is negative (and large again) inside the window, so that the mentioned ratio equals $-3^{1/3}\langle N_0(x, x_0, t)\rangle^{1/6}$. This behaviour is illustrated below on the example of the discrete model (see figure 3).

Equation (2.21) explicitly shows the self-averaging at the late stage of branching. We did not study this self-averaging for multiple (triple, etc) fragmentation processes—its existence for slowing fragmentation was proven mathematically by Filippov [2].

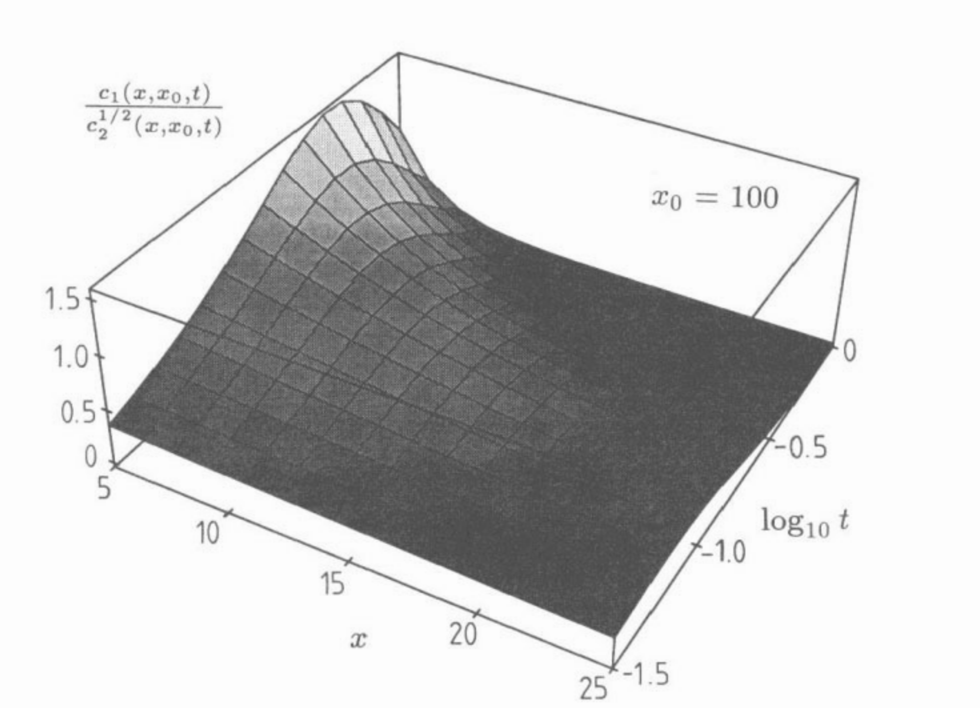

Figure 1. Solution of (2.12) for the second moment with kernel (2.1) (at $\beta = 0$) plotted as the signal-to-noise ratio. For the analytic expression of the second moment, see appendix C.

2.2. Below shattering, $\beta < -1$

In this case fluctuations dominate. To show this we may evaluate the triple integral (2.12) using the asymptotical solution (2.5). Some study shows that the integral gains its value at $z \sim y$, $t' \sim t$, $y \sim x_0$ if we assume $tx_0^{\beta+1} \sim 1$, about the maximum of occupation for a given x. The result is

$$\langle N^2(x, x_0, t \sim x_0^{-\beta-1})\rangle \sim x^{-2\beta-6} x_0^{2\beta+4} \sim \langle N(x, x_0, t)\rangle^2 \qquad (2.22)$$

and the second cumulant is of the same order. One concludes that the signal-to-noise ratio is of order of unity here. The situation changes at $\beta < -2$, where $\langle N(x, x_0, t)\rangle \ll 1$. Again the first term on the right-hand side of (2.12) (and the very first splitting) becomes important and fluctuations dominate strongly. The signal-to-noise ratio is as small as $(x_0/x)^{(\beta+2)/2}$. Finally, the mass of dust (which is defined as the initial mass minus the mass of all fragments with essentially positive mass, $1 - \int_0^{x_0} \mathrm{d}x\, N(x, x_0, t)$) is subject to the fluctuations mentioned.

3. The steady-state regime

As has been shown the fluctuations may be large and any observation makes sense only if the results are expressed in terms of the values averaged over a number of independent repetitions, larger than c_2/c_1^2. In many physical systems (cosmic showers, electron, phonon transport) one deals instead with a source of incoming 'particles' having the initial mass (energy) x_0 and collects the outgoing fragments after many breakups. The question now concerns establishing the steady state in such an experiment and the minimum source intensity required to suppress the fluctuations which may still be present in the steady state.

We shall introduce the simple procedure (analogous to [15]) which should provide such an averaging (steady state). At the very first state (x_0) there is assumed to be a constant discrete source of new 'particles', which produces new cascades with a time delay τ between each two; this time is a random variable distributed with the Poisson probability $P(\tau) = e^{-\tau/\bar{\tau}}$. The parameter $\bar{\tau}$ is inversely proportional to the source intensity. In our system the evolution of each particle is independent, and the logarithmic generating function $Z = \ln(G)$ is an additive variable. If there exists a steady-state regime, these functions should be equal to

$$Z(x, x_0; z) = \sum_{k=0}^{\infty} Z\left(x, x_0, t - \sum_{m=0}^{k} \tau_m; z\right). \tag{3.1}$$

Here τ_m denotes the mth realization of the random value τ. We can average the above-written sum over the distribution $P(\tau)$ to find

$$Z(x, x_0; z) = \frac{1}{\bar{\tau}} \int_0^{\infty} dt\, Z(x, x_0, t; z). \tag{3.2}$$

Clearly, the same result is valid for the average properties of the same system with a periodic source where the period is $\bar{\tau}$. In the steady-state regime it is natural to work with cumulants (2.10), which are connected with the time-dependent cumulants due to (3.2)

$$c_n(x, x_0) = \frac{1}{\bar{\tau}} \int_0^{\infty} dt\, c_n(x, x_0, t). \tag{3.3}$$

The corresponding equations for moments can be derived from here. Note that the equations for the time-dependent cumulants (obtainable from (2.8)) allow one to reduce the integration in (3.3) to an integral equation, whose usage may sometimes be easier. For example, by integrating the backward kinetic equation over time from 0 to ∞ we find the integral equation for c_1 ($= \langle N \rangle$)

$$c_1(x, x_0) \int_0^{x_0} dy\, K(y, x_0) = 2 \int_x^{x_0} dy\, K(y, x_0) c_1(x, y). \tag{3.4}$$

The equation for $c_2(x, x_0)$ can be derived from (2.12), etc. The criterion on the minimum source intensity to have the signal-to-noise ratio large in the steady-state regime is given by $1/\bar{\tau} \gg c_2/c_1^2$ and depends upon x, x_0. Below we set the time-delay constant $\bar{\tau}$ to unity—it can be easily retrieved by rescaling cumulants.

The existence of the steady state is now connected with the convergence of the integrals (3.3). In principle it could happen that some of these integrals converge (that for c_1, for example) but others do not. It does not take place for the linear systems considered here, since the maximum occupation occurs at finite time. Nevertheless, we shall find that the *time* required to establish the steady state depends upon the order of the cumulants of distribution—the higher the order, the longer it may take for the corresponding integral to converge. The distribution under study is far from Gaussian, so that high cumulants may be of interest. Usually it is instructive enough to study the convergence for low-order cumulants c_n, $n = 1, 2, 3$.

To get some experience concerning the properties of different quantities it is useful to investigate in some detail the simple Yule–Ferry discrete model [11] solved in the formulation following [13]. Namely, consider a set of generations $n \geqslant 0$. One initial particle ($n = 0$) decays creating two particles in the first generation (breakup to two halves); each particle of this generation ($n = 1$) decays independently creating two particles in the second generation and so on. Let the probability $W(N, n, t|1, 0, 0)$ describe the presence of N particles in generation n at time t given there was one particle at the very beginning (cf section 2). The generating function for W is

$$G_n(t; z) = \sum_N W(N, n, t|1, 0, 0) \exp(zN) \tag{3.5}$$

where we omit the indexes describing our initial condition. It obeys the equation (cf (2.8))

$$\partial_t G_n(t; z) = -G_n(t; z) + G_{n-1}^2(t; z) . \tag{3.6}$$

It follows that the equations for the first three moments of W are

$$\begin{aligned} \partial_t \langle N_n \rangle &= -\langle N_n \rangle + 2\langle N_{n-1} \rangle \\ \partial_t \langle N_n^2 \rangle &= -\langle N_n^2 \rangle + 2\langle N_{n-1}^2 \rangle + 2\langle N_{n-1} \rangle^2 \\ \partial_t \langle N_n^3 \rangle &= -\langle N_n^3 \rangle + 2\langle N_{n-1}^3 \rangle + 6\langle N_{n-1}^2 \rangle \langle N_{n-1} \rangle \end{aligned} \tag{3.7}$$

where $\langle N_n^k(t) \rangle = \partial^k G_n(t; z)/\partial z^k{}_{|z=0}$. Equations (3.7) can be recurrently solved by using Laplace transform in time. Solutions for $N_{1,n}$ and $N_{2,n}$ have been obtained in [13] by another technique. The Laplace transform with the initial condition $\langle N_n(0) \rangle = \delta_{n,0}$ gives us the Poisson-type solution for the occupation numbers

$$\langle N_n(t) \rangle = \frac{2^n}{n!} t^n \mathrm{e}^{-t} . \tag{3.8}$$

The maximum mean occupation at a given site n occurs after a time $t = n$. We would like to study how fluctuations affect this simple picture. The same transform gives the (rather cumbersome) solution for the mean square occupation number at a given site:

$$\begin{aligned} \langle N_n^2(t) \rangle = \frac{2^n}{n!} t^n \mathrm{e}^{-t} + \mathrm{e}^{-t} \sum_{m=1}^{n} \sum_{l=0}^{n-m} \frac{(-1)^l 2^{n+m-1} (2m+l-2)!}{[(m-1)!]^2 (n-m-l)!(l)!} t^{n-m-l} \\ + \mathrm{e}^{-2t} \sum_{m=1}^{n} \sum_{l=0}^{2m-2} \frac{(-1)^{n-m+l} 2^{n+m-1} [2(m-1)]!(n-m+l)!}{(n-m)![(m-1)!]^2 (2m-l-2)!(l)!} t^{2m-l-2} . \end{aligned} \tag{3.9}$$

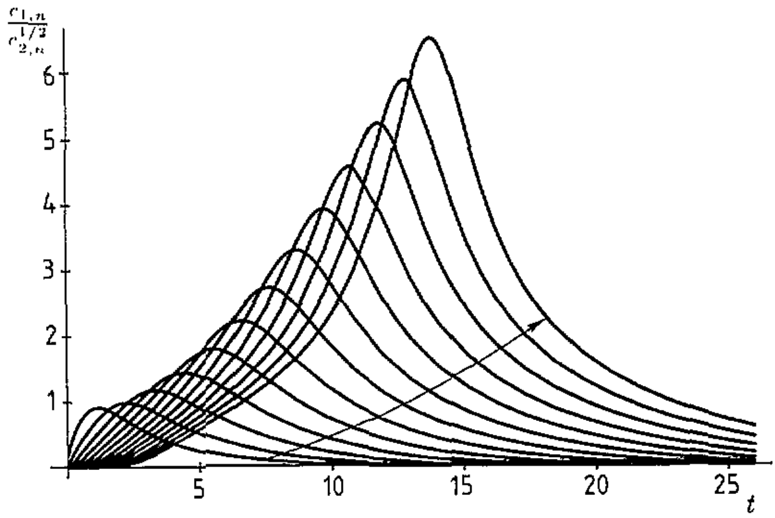

Figure 2. Signal-to-noise ratios at different sites $n \in [1, 13]$ versus time in the simple cascade Yule–Ferry model. The arrow shows the increase of the generation number.

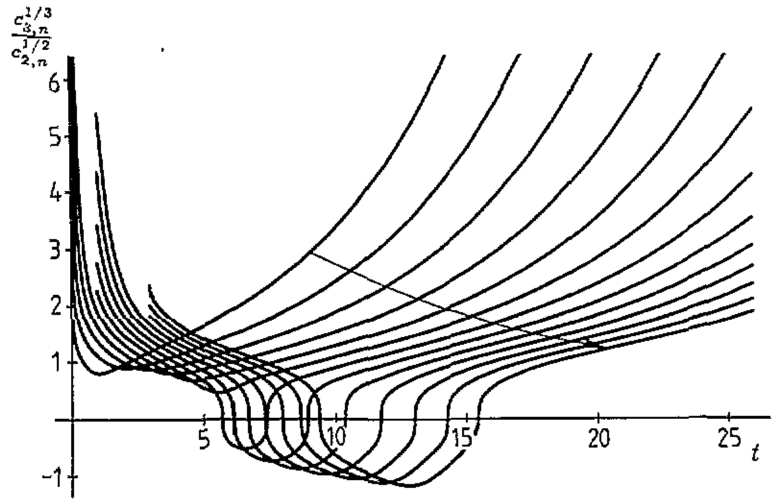

Figure 3. Cubic root of third cumulant divided by square root of second cumulant—the non-Gaussian property of occupation numbers. Sites and times are as in figure 2.

The signal-to-noise ratio $c_1/c_2^{1/2}$ is shown in figure 2. The arrow indicates the increase of n. In the discrete model, fluctuations are also small near the maximum of occupation for $n \gg 1$. Figure 3 shows the ratios $c_3^{1/3}/c_2^{1/2}$ showing that fluctuations in our simple cascade are entirely non-Gaussian and non-symmetric. The behaviour of the third cumulant is quite rich and analogous to that found in section 2. The asymmetry is almost always large and is initially right-sided, then left-sided in the window, and finally right-sided again.

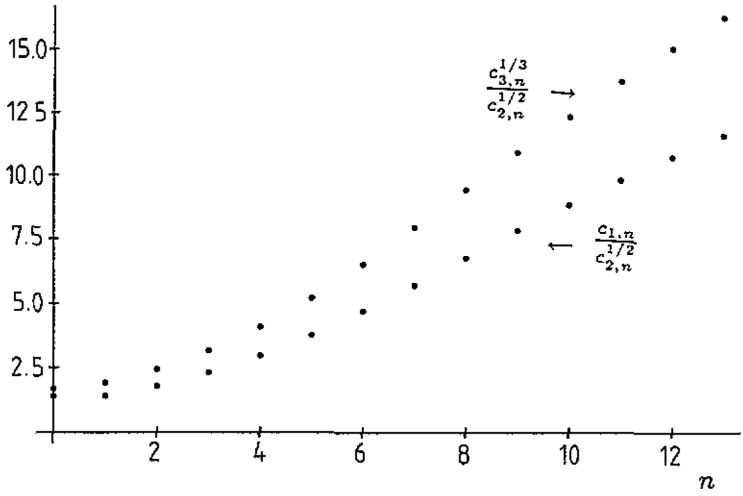

Figure 4. Steady-state occupancies in the simple cascade model at different generations. The source intensity is unity.

We return to the steady-state regime now. The steady-state occupation number for the generation n follows from (3.8) and (3.3):

$$c_{1,n} = \langle N_n \rangle = \int_0^\infty \mathrm{d}t \, \frac{(2t)^n \mathrm{e}^{-t}}{n!} = 2^n \,. \tag{3.10}$$

As for the second cumulant, it is easier to make use of (3.7) and the equation for the second cumulant to find a recurrence relation for the steady-state cumulants

$$c_{2,n} = 2c_{2,n-1} + \int_0^\infty \mathrm{d}t \, (2c_{1,n-1} - c_{1,n})^2 = 2c_{2,n-1} + \frac{(2n-2)!}{n!(n-1)!} \,. \tag{3.11}$$

The solution is given by the expression

$$c_{2,n} = 2^{n-1} + \sum_{k=0}^{n-1} \frac{2^k (2k)!}{k!(k+1)!} \,. \tag{3.12}$$

The ratios $c_{1,n}/c_{2,n}^{1/2}$ and $c_{3,n}^{1/3}/c_{2,n}^{1/2}$ are shown in figure 4. The convergence of time integrals (3.3) for c_1, c_2, c_3 is shown in figure 5. We conclude that the steady-state regime is well defined in this model and fluctuations become less and less important with the generation number. However, the distribution is again non-symmetric around its maximum. It is also interesting that the time required for the third cumulant to converge is larger than that for c_1, c_2.

Keeping the above results in mind let us return to continuum models with the random source acting at the initial point x_0 (section 2). The stationary solution of the kinetic equation can be found by either direct integration of its time-dependent solution (2.3) in (3.3), or by solving the integral equation (3.4):

$$x_0 c_1(x, x_0) = 2 \int_x^{x_0} \mathrm{d}y \, c_1(x, y) \tag{3.13}$$

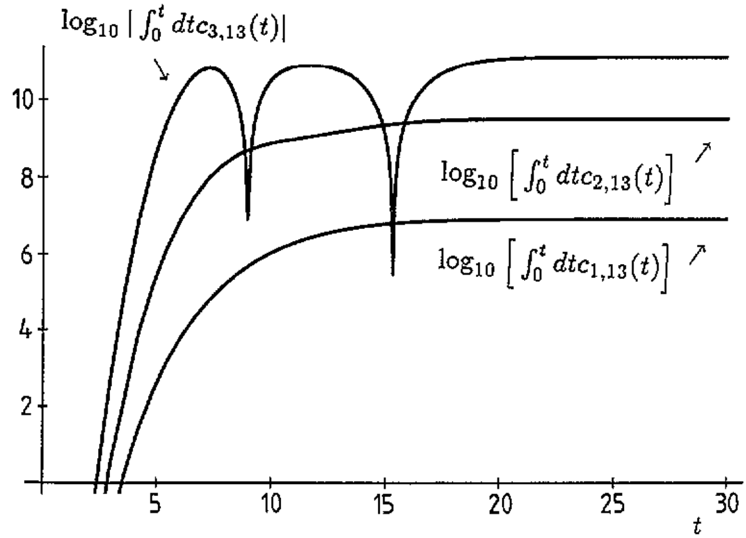

Figure 5. Convergence of the first, second and third cumulants to their steady-state values from figure 4 at the site (generation) $n = 13$.

which can be converted into a differential equation. The solution is

$$c_1(x, x_0) = x_0^{-\beta-1}\delta(x - x_0) + 2x_0 x^{-\beta-3}. \tag{3.14}$$

Note that the total mass of all finite fragments is infinite if $\beta \geqslant -2$, and it would take an increasingly long time, $t \sim \max(x^{-\beta-1}, x_0^{-\beta-1})$, for establishing such a distribution for small fragment sizes.

Integrating the expression for the second cumulant following from (2.21) one finds at $\beta > -1$, $x \ll x_0$ that

$$c_2(x, x_0) = c_1(x, x_0) \tag{3.15}$$

and, recalling the source intensity $1/\bar{\tau}$, the steady-state signal-to-noise ratio is large at

$$1/\bar{\tau} \gg x^{\beta+3}/x_0. \tag{3.16}$$

In the region $-2 < \beta < -1$, $x \ll x_0$ the estimate (2.21) gives the x-independent ratio $c_1(x, x_0)/c_2^{1/2}(x, x_0) \sim x_0^{-1/2}$, and $1/\bar{\tau} \gg x_0$. At $\beta < -2$ (3.15) and therefore (3.16) are again valid if $x \ll x_0$.

4. Connection with the combinatorial approach

We discuss here the connection between two different methods to study fragmentation processes which exist in the literature [3, 16, 17, 6]. The alternative to the kinetic method is the approach (we call it combinatorial) to declare that a given definite amount of fragments $\overline{N}$ is found in an experiment [6]. The problem would be to describe what is the mass distribution of these observed fragments and then to compare it with the measurements. The solution leads to combinatorial analysis of how many ways exist to get the same number $\overline{N}$ and what are their probabilities. The time passed does not play any role in the problem.

Generally speaking, the combinatorial approach is artificial (a second experiment would give another number $\overline{N}'$) and does not reflect the proper behaviour of the system. While both approaches sample the phase space of the problem, the kinetic approach takes into account all the possible states, while the combinatorial one only takes care of those which have the same number $\overline{N}$ and it is clear that only the mean-field properties can be captured. Fluctuations are different.

Indeed, Grady [6] implemented the combinatorial approach to study fragmentation in the simple case of uniform breakup ($\beta = 0$ in terms of kernel (2.1)). The distribution

$$\int_x^{x_0} \mathrm{d}x' \langle N(x', x_0, \overline{N}) \rangle = \overline{N} \left(\frac{x_0 - x}{x_0} \right)^{\overline{N}-1} \tag{4.1}$$

was obtained in the case of a continuous finite body. For the same problem the solution (2.3) for $\beta = 0$ implies

$$\int_x^{x_0} \mathrm{d}x' \langle N(x', x_0, t) \rangle = (1 - tx + tx_0)\mathrm{e}^{-tx} \, . \tag{4.2}$$

We expect the combinatorial approach to become meaningful only at large times or, equivalently, at small sizes ($x \ll x_0$). Indeed, from (4.1) at $\overline{N} \gg 1$, $x \ll x_0$ the asymptotic form follows $\overline{N} \exp(-x\overline{N}/x_0)$, which agrees with the asymptotic form $tx_0 \exp(-tx)$ for (4.2) if we establish the physically obvious connection $t = \overline{N}/x_0$. Above shattering there is an equivalence between the regions of validity of combinatorial and similarity solutions.

Below we shall address the abovementioned connection for the final or steady-state regimes. As time proceeds so does the fragmentation and finally a collection of objects of the smallest possible size would result. At this stage the system arrives at the final deterministic state. However, if one adds an additional rule that some of the intermediate fragments cannot proceed further, the final state may be distributed. Such an example is given by the random breaking of an interval studied by Derrida and Flyvbjerg [5]. In each binary breakup one of the two fragments is not allowed to proceed further and contributes to the final distribution. The other fragment breaks up and the scheme repeats. The model has very rich properties.

Although time does not appear in this problem we shall show below how to introduce it and derive the Derrida and Flyvbjerg equation for the final distribution. Correlators of the distribution can also be obtained by the same method. Let $P(N, x, t|x_0, 0)$ denote the probability density to have N fragments of size (mass, energy) x at time t given that the process starts with one 'particle' at $x = x_0$, $t = 0$. By analogy with (2.6) the function P obeys the following (backward) equation

$$\partial_t P(N, x, t|x_0, 0) = -P(N, x, t|x_0, 0) \int_0^{x_0} \mathrm{d}y \, K(y, x_0)$$
$$+ \tfrac{1}{2} \int_0^{x_0 - x} \mathrm{d}y \, K(y, x_0) P(N, x, t|x_0 - y, 0)$$

$$+\tfrac{1}{2}\int_x^{x_0} dy\, K(x_0-y,x_0)P(N,x,t|y,0)$$
$$+\tfrac{1}{2}\Theta\left(\frac{x_0}{2}-x\right)(1-\delta_{N,0})[K(x,x_0)+K(x_0-x,x_0)]$$
$$\times[P(N-1,x,t|x_0-x,0)-P(N,x,t|x_0-x,0)]$$
$$+\tfrac{1}{2}\Theta\left(x-\frac{x_0}{2}\right)\delta_{N,1}[K(x,x_0)+K(x_0-x,x_0)][1-P(N,x,t|x_0-x,0)]\,. \tag{4.3}$$

Here the first integral on the right-hand side deals with the case when no breakup occurs at the very first state. The second and third integrals describe breakup when the first fragments of size y and x_0-y contribute to the final distribution, while the complementary one proceeds further and may contribute to P later. There are special cases when the fragment y (or x_0-y) arrives exactly at the point x and contributes explicitly to P. If $x>x_0/2$, this is the only way for the point x to become occupied (multiplier $\delta_{N,1}$), else the N argument of the P-function is changed to $N-1$. The non-integral P-terms with minus signs take care of the situation when the exact arrival is contained in the integral terms, and should be excluded.

The equation for the generation function defined by analogy with (2.7) reads

$$\partial_t G(x,t|x_0,0;z) = -G(x,t|x_0,0;z)\int_0^{x_0} dy\,\bar{K}(y,x_0)$$
$$+\int_0^{x_0} dy\,\bar{K}(y,x_0)G(x,t|y,0;z)$$
$$+\bar{K}(x,x_0)(e^z-1)\left[\Theta\left(\frac{x_0}{2}-x\right)G(x,t|x_0,0;z)+\Theta\left(x-\frac{x_0}{2}\right)\right] \tag{4.4}$$

where $\bar{K}(x,y)=[K(x,y)+K(y-x,y)]/2$. For the first moment (occupation numbers) we have

$$\partial_t\langle N(x,t|x_0,0)\rangle = -\langle N(x,t|x_0,0)\rangle\int_0^{x_0} dy\,\bar{K}(y,x_0)$$
$$+\int_0^{x_0} dy\,\bar{K}(y,x_0)\langle N(x,t|y,0)\rangle+\bar{K}(x,x_0)\,. \tag{4.5}$$

Omitting the time derivative we get the integral equation for the final distribution. If we consider the self-similar kernels $\bar{K}(x,x_0)=K(x/x_0)$, then the final distribution also becomes self-similar, $\langle N(x,x_0)\rangle=\langle N(x/x_0)\rangle$. Introducing the new variable of integration $y=x_0-x$, we get the equation equivalent to that obtained in [5]:

$$\langle N(x)\rangle=\frac{1}{x_0}K(x)+\int_0^{1-x} dy\,K(y)\left\langle N\left(\frac{x}{x_0-y}\right)\right\rangle\,. \tag{4.6}$$

Note, that the kernel $K(x)$ has to be necessarily symmetric as is shown above. This means that the solutions obtained in [5] for a non-symmetric kernel do not apply to the considered fragmentation process. Other characteristics of randomly broken objects [5] can be studied by this approach.

5. Fluctuations in phonon decay

In this section we deal with the fragmentation process for the example of phonons. In order to be specific we consider the process of the phonon frequency down-conversion studied by Kazakovtsev and Levinson [7]. The physical situation is the following. A population of hot phonons is created in a semiconducting or insulating sample. Somewhere inside there is a device which measures (at least in principle) the energy distribution of the arriving low-frequency phonons. The energy evolution of each phonon ϵ_0 is the familiar process of anharmonic splitting into two daughter phonons, ϵ, $\epsilon_0 - \epsilon$; the conventional approximation for the kernel reads [7]

$$K(\epsilon, \epsilon_0) = A\epsilon^2(\epsilon_0 - \epsilon)^2 . \tag{5.1}$$

If the occupation numbers (here in the quantum sense) are smaller than unity, the phonon cascade is linear, each phonon is independent. As for the space propagation, it can easily be included since the energy evolution is independent of that in space, and the problem is to 'spread' the distribution in space in accordance to its energy content.

We may therefore immediately make use of (2.11) which now reads

$$\begin{aligned}\partial_t \langle N^2(\epsilon, t|\epsilon_0, 0)\rangle = &-\langle N^2(\epsilon, t|\epsilon_0, 0)\rangle \int_0^{\epsilon_0} \mathrm{d}\epsilon'\, K(\epsilon', \epsilon_0) \\ &+ 2c \int_{\epsilon}^{\epsilon_0} \mathrm{d}\epsilon'\, K(\epsilon', \epsilon_0)\langle N^2(\epsilon, t|\epsilon', 0)\rangle \\ &+ 2c \int_{\epsilon}^{\epsilon_0 - \epsilon} \mathrm{d}\epsilon'\, K(\epsilon', \epsilon_0)\langle N(\epsilon, t|\epsilon', 0)\rangle\langle N(\epsilon, t|\epsilon_0 - \epsilon', 0)\rangle\end{aligned} \tag{5.2}$$

where c is a number resulting from the presence of density of states, $\rho(\epsilon) \propto \epsilon^2$. We make use of the similarity solution of the kinetic equation, which has the form [7]

$$\langle N(\epsilon, r, t|\epsilon_0, 0, 0)\rangle = C\epsilon^{-4} f\left(\frac{t}{\tau(\epsilon)}\right) . \tag{5.3}$$

Here the normalization to one phonon is assumed, $C = \epsilon_0^3/\rho(\epsilon_0)V$ with V being the normalization volume, and $\tau^{-1}(\epsilon)$ is the inverse phonon lifetime, which is defined as $\int_0^{\epsilon} \mathrm{d}\epsilon'\, K(\epsilon', \epsilon)$. We deal here with the fragmentation above shattering, and may immediately apply the result (2.21) of section 2

$$\langle N^2(\epsilon, t|\epsilon_0, 0)\rangle = \langle N(\epsilon, t|\epsilon_0, 0)\rangle^2 + \langle N(\epsilon, t|\epsilon_0, 0)\rangle \tag{5.4}$$

and corresponding formulae for higher cumulants. Here $\langle N\rangle$ has the meaning of the number of phonons in a given 'box' v in space and $\rho(\epsilon)\Delta\epsilon$ in energy space. As has been said in section 2, the result (5.4) shows that fluctuations are not large around the maximum of the similarity solution (5.3), where $v\rho(\epsilon)\Delta\epsilon\langle N\rangle \gg 1$ and are given simply by the square root of this average number of phonons in the box. These fluctuations are entirely non-Gaussian in the same sense as we have explained above.

6. Conclusion

We have presented the application of the kinetic approach of backward equations to the study of different systems evolving along tree-like paths in the phase space (Bethe lattice, Cayley tree and ultrametric manifold are the other names used). For the systems below the shattering threshold our results show that fluctuations dominate and the mean-field solution can be 'measured' only by using the given large number of repetitions. For larger systems, the fluctuations are exceedingly strong *if* they dominate. We have shown that the similarity solution is the result of self-averaging along the branching cascade if it slows down. The steady-state regime with an external source has been considered, which may suppress fluctuations even below shattering due to the presence of many particles proceeding simultaneously. We have also shown that the combinatorial and kinetic approaches provide equivalent results only for very simple models in the mean-field approximation. Exact solutions have been given for some models. The results obtained for the slowing-down fragmentation have been used to study the phonon frequency-down conversion.

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Appendix A

We present here the formal derivation of our basic equation of (2.6) type for the discrete Yule–Ferry cascade model. The continuous case of (2.6) can be treated in an analogous way. Let us introduce the vector of state $\mathbf{N} = \{N_0, N_1, N_2, \ldots, N_n, \ldots\}$ which describes the occupation numbers at the sites $0, 1, 2, \ldots, n, \ldots$. The probability of the system having a given state $\mathbf{N}$ at time t provided it has had the state $\mathbf{N}'$ at time t' is denoted as $P(\mathbf{N}, t|\mathbf{N}', t')$. It is connected with the 'short' description $W(N, n, t|1, 0, 0)$ due to the reduction relation

$$W(N, n, t|1, 0, 0) = \sum_{\text{all } N_k,\ k \neq n} P(\mathbf{N}, t|\mathbf{I}, 0) \tag{A1}$$

where $\mathbf{I} = \{1, 0, 0, \ldots\}$. The probability P obeys the forward and backward Chapman–Kolmogorov equations [11]. The forward equation is equivalent here to the master equation and can be written as

$$\partial_t P(\mathbf{N}, t|\mathbf{N}', t') = \sum_{\text{all } \mathbf{N}''} [w(\mathbf{N}'' \to \mathbf{N}) P(\mathbf{N}'', t|\mathbf{N}', t') - w(\mathbf{N} \to \mathbf{N}'') P(\mathbf{N}, t|\mathbf{N}', t)] \tag{A2}$$

where $w(\mathbf{N} \to \mathbf{N}')$ is the transition rate from the state **N** to the state **N'**. In our case it is equal to N_n for the transition when the occupation number N_n diminishes by one and the next generation occupation number N_{n+1} increases by two, and is zero otherwise. We apply the reduction formula (A1) attempting to derive a closure for the 'short' description and immediately find that the sum

$$\sum_{N_{n-1}} (N_{n-1}+1) P(\{\ldots, N_{n-1}+1, N_n-2, \ldots\}, t|\ldots) \tag{A3}$$

which appears in the first term on right-hand side of (A2) where summation over all N_k have been performed except for $k = n-1$, does not reduce to the short description, because we cannot perform the last summation in terms of W functions. One can see that the forward time equation always generates the higher-order correlators.

The situation is quite different for the backward Chapman–Kolmogorov equation,

$$\partial_{t'} P(\mathbf{N}, t|\mathbf{N}', t') = \sum_{\text{all } \mathbf{N}''} w(\mathbf{N}' \to \mathbf{N}'') \left[P(\mathbf{N}, t|\mathbf{N}', t') - P(\mathbf{N}, t|\mathbf{N}'', t) \right] . \tag{A4}$$

Applying the summation (A1) and the above-mentioned transition rate w we derive the equation analogous to that of Williams [12],

$$\partial_{t'} W(N, n, t|1,0,0) = W(N, n, t|1,0,0) - W(N, n, t|2,1,0) \tag{A5}$$

which with the help of the identity

$$W(N, n, t|2,1,0) = \sum_{N'} W(N', n, t|1,1,0) W(N-N', n, t|1,1,0) \tag{A6}$$

the change of $\partial_{t'} = -\partial_t$, shifting the argument $(N, n, t|1,1,0) \to (N, n-1, t|1,0,0)$, and using the generation function representation (3.5) gives (3.6).

Appendix B

In the case of more than binary fragmentation the generalization of (2.8) is straightforward. Let us first write down the equation for the triple fragmentation; the triple kernel $K_3(x, y, x_0)$ describes the breakup rate of the initial particle x_0 into three fragments $x, y, x_0 - x - y$, the corresponding symmetry is assumed:

$$\begin{aligned}
\partial_t G(x, x_0, t; z) = & -G(x, x_0, t; z) \int \mathcal{D} \\
& + \int \mathcal{D} G(x, x', t; z) G(x, y', t; z) G(x, x_0 - x' - y', t; z) \\
& + \int \mathcal{D} G(x, y', t; z) G(x, x_0 - x' - y', t; z) \\
& + \int \mathcal{D} G(x, x', t; z) G(x, x_0 - x' - y', t; z) \\
& + \int \mathcal{D} G(x, x', t; z) G(x, y', t; z) \\
& + \int \mathcal{D} G(x, x', t; z) + \int \mathcal{D} G(x, y', t; z) + \int \mathcal{D} G(x, x_0 - x' - y', t; z)
\end{aligned} \tag{B1}$$

where $\mathcal{D} = \mathrm{d}x'\mathrm{d}y'K_3(x', y', x_0)$. The integration regions should be organized in such a way that all the second arguments (of G-functions being integrated) have to lie between x and $x_0 - x$ altogether. For example, the integral containing the triple product of G-functions is to be performed over the region $\{x' \geqslant x, y' \geqslant x, x' + y' \leqslant x_0 - x\}$. All the integration regions are supposed to be positive.

It is simple to modify (B1) for k-pieces breakup. There is the main k-product of all G-functions and subsequent terms describing that only some of the k-pieces (those which have sizes from x to x_0 in all the possible combinations) can contribute to the distribution at the point x.

The next step is to take the sum of all these equations for arbitrary k and finally compute the time derivative of the generation function,

$$\partial_t G(x, x_0, t; z) = \partial_t \sum_{k=2}^{\infty} G_k(x, x_0, t; z) . \tag{B2}$$

Although this equation is rather cumbersome, the corresponding equations for the moments simplify considerably. For example, studying the first moment, we have to take the first derivative of (B2) with respect to z. This gives the backward kinetic equation

$$\begin{aligned}\partial_t\langle N(x, x_0, t)\rangle = -\langle N(x, x_0, t)\rangle \int_0^{x_0} \mathrm{d}y\, \mathcal{K}_{\langle N\rangle}(y, x_0) \\ + \int_x^{x_0} \mathrm{d}y\, \mathcal{K}_{\langle N\rangle}(y, x_0)\langle N(x, y, t)\rangle\end{aligned} \tag{B3}$$

with the *first* effective kernel

$$\begin{aligned}\mathcal{K}_{\langle N\rangle}(x, x_0) = 2K_2(x, x_0) + 3\int_0^{x_0} \mathrm{d}y_1\, K_3(x, y, x_0) + \cdots \\ + (n+2)\int\cdots\int \mathrm{d}y_1\cdots\mathrm{d}y_n\, K_{n+2}(x, y_1, \ldots, y_n, x_0) + \cdots\end{aligned} \tag{B4}$$

where the integration is performed over the region of non-negative $\{y_1, \ldots, y_n\}$ satisfying the condition $x + y_1 + \cdots + y_n \leqslant x_0$.

The second moment $\langle N^2\rangle$ obeys the following equation:

$$\begin{aligned}\partial_t\langle N^2(x, x_0, t)\rangle = -\langle N^2(x, x_0, t)\rangle \int_0^{x_0} \mathrm{d}y\, \mathcal{K}_{\langle N\rangle}(y, x_0) \\ + \int_x^{x_0} \mathrm{d}y\, \mathcal{K}_{\langle N\rangle}(y, x_0)\langle N^2(x, y, t)\rangle \\ + \int_x^{x_0-x} \mathrm{d}y\, \mathcal{K}_{\langle N^2\rangle}(y, x_0)\langle N(x, y, t)\rangle\langle N(x, x_0 - y, t)\rangle\end{aligned} \tag{B5}$$

with the *second* effective kernel

$$\begin{aligned}\mathcal{K}_{\langle N^2\rangle}(x, x_0) = 2K_2(x, x_0) + 6\int_0^{x_0} \mathrm{d}y_1\, K_3(x, y, x_0) + \cdots \\ + (n+1)(n+2)\int\cdots\int \mathrm{d}y_1\cdots\mathrm{d}y_n\, K_{n+2}(x, y_1, \ldots, y_n, x_0) + \cdots\end{aligned} \tag{B6}$$

having the same rules for the integration region as (B4). Therefore, the power-law assumption [3] for the first argument x of the kernel $\mathcal{K}_{\langle N\rangle}(x, x_0)$ does not specify higher-order effective kernels, except for the case of binary fragmentation. An independent power-law assumption can be made for $\mathcal{K}_{\langle N^2\rangle}(x, x_0)$ following Filippov.

Appendix C

The solution for the second moment $\langle N^2(x, x_0, t)\rangle$ in the case $\beta = 0$ can be obtained from (2.12) by inserting into it the first moment (2.3) which now reads

$$\langle N(x, x_0, t)\rangle = \mathrm{e}^{-tx_0}\delta(x - x_0) + t(2 + t(x_0 - x))\mathrm{e}^{-tx} . \tag{C1}$$

The following expression results by applying a symbolic manipulation program:

$$\langle N^2(x, x_0, t)\rangle = \langle N(x, x_0, t)\rangle + t^2(6 - 12tx + 4t^2x^2 + 6tx_0 - 4t^2xx_0 + t^2x_0^2)\mathrm{e}^{-2tx} \tag{C2}$$

where $x \leqslant x_0/2$. In the region $x > x_0/2$ the second moment is not affected by the inhomogeneity in (2.11) and is just equal to the first moment. Note that the continuum limit corresponds to $x_0 \gg 1$. However, at any finite x_0 there are discontinuities in $\langle N^2(x, x_0, t)\rangle$, i.e. at $x = x_0, x_0/2$ the latter one becomes small at large x_0.

If we subtract $\langle N(x, x_0, t)\rangle^2$ from $\langle N^2(x, x_0, t)\rangle$ given by (C2) (or simply by $\langle N(x, x_0, t)\rangle$ at $x > x_0/2$) and integrate over time, we obtain the steady-state value of the second cumulant

$$c_2(x, x_0) = \begin{cases} 2x_0/x^3 - 1/4x^3 - 3x_0^2/4x^5 & x > x_0/2 \\ 2x_0/x^3 - 1/4x^3 - 3x_0/4x^4 & x < x_0/2 \end{cases} \tag{C3}$$

where the first term in both of the expressions (C3) results from the integration of $\langle N(x, x_0, t)\rangle$ over time.

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